Abel-Jacobi Condition for Quadrilateral Mesh Generation

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Thanks

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Collaborators

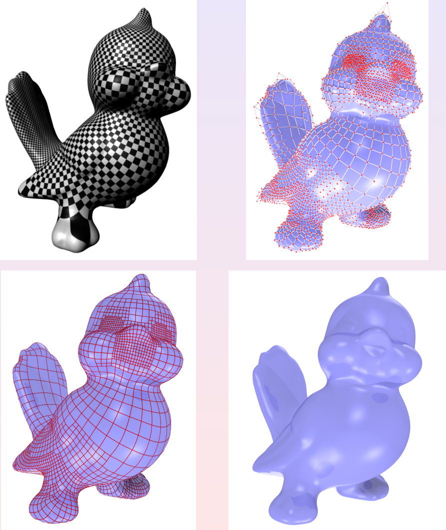
This is the joint work with Na Lei. Related works are collaborated with Prof. Shing-Tung Yau, Jian Sun, Tianqi Wu, Jorg Peters, Tom Hughes, Tom Sederberg and many others.

Outline

- Motivation;
- Quad-mesh and Meromorphic quartic differential;
- Holomorphic one-form;
- Holomorphic quadratic differential;
- Meromorphic quartic differential;
- Conclusion;

Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.



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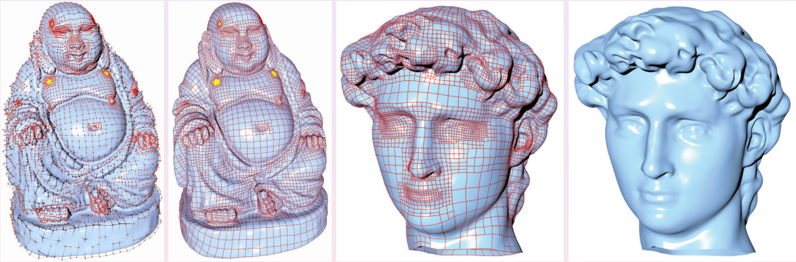
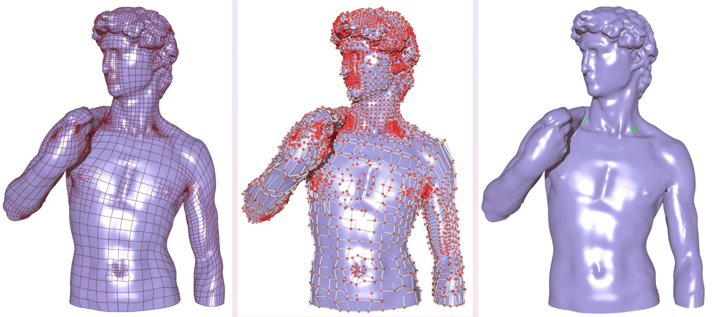
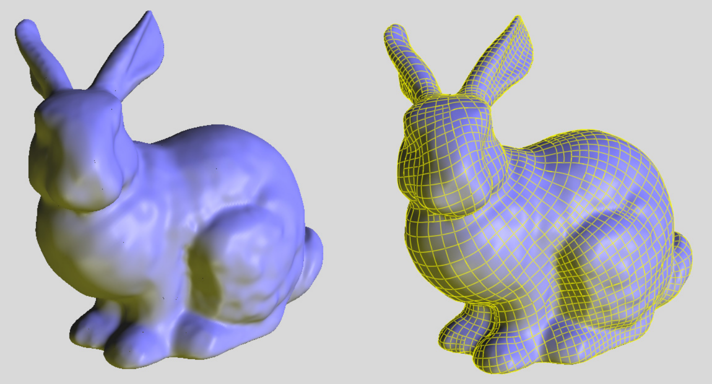


Polygonal mesh to spline, control net and the knot structure.



Manifold Spline



Quad-Mesh

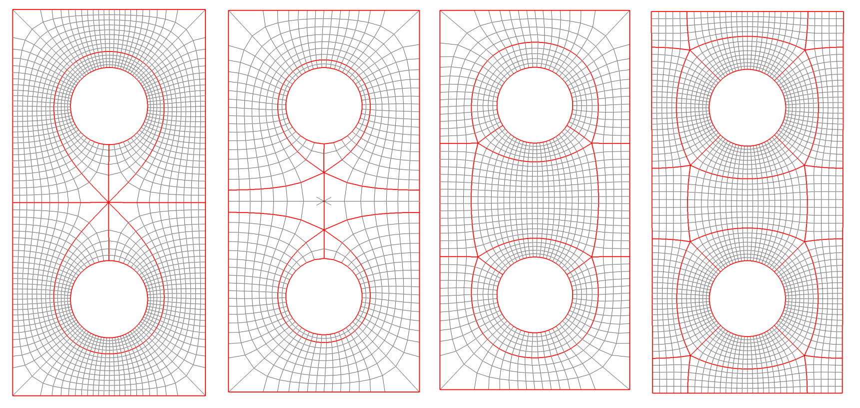


Figure: Quad-meshes with different number of singularities.

Quad-Mesh

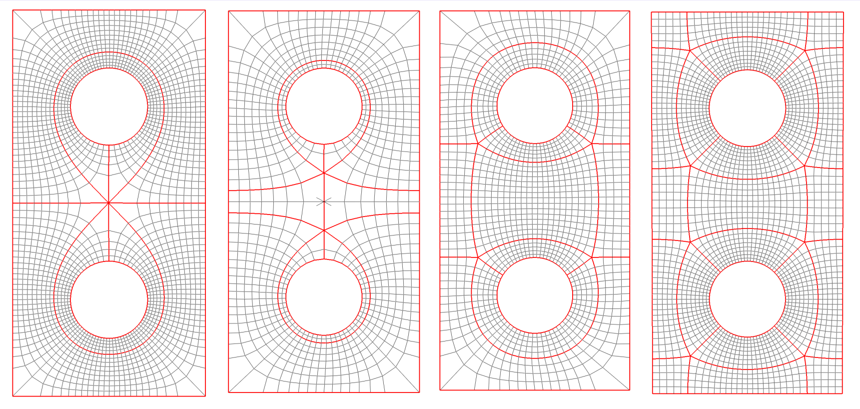


Figure: A holomorphic 1-form, a quadratic differential, and a meromorphic quartic differential.

Quad-Mesh

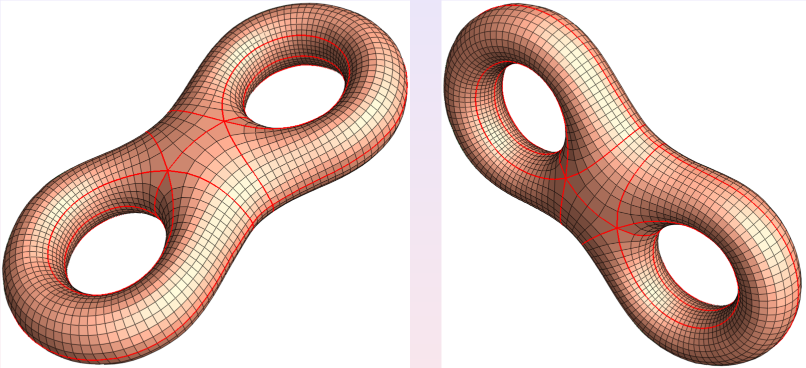


Figure: Quad-meshes with different number of singularities, a holomorphic quadratic differential.

Quad-Mesh

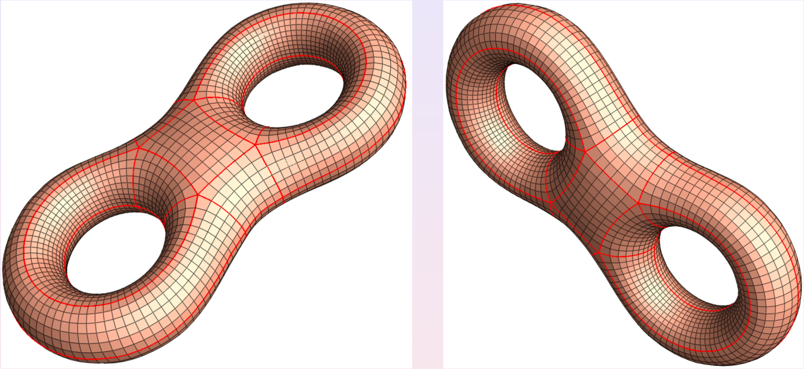
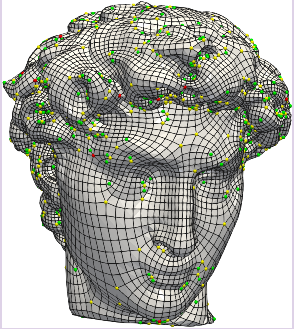
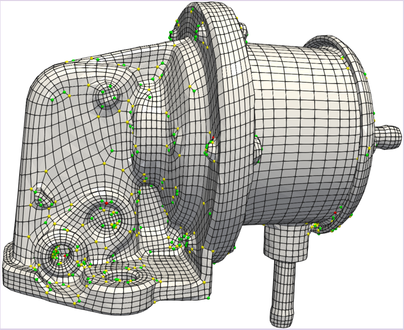


Figure: Quad-meshes with different number of singularities, a meromorphic quartic differential.

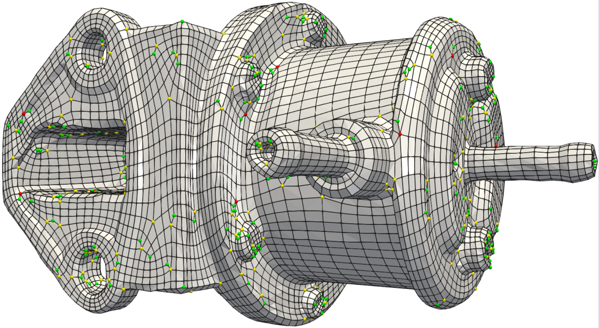
Singularities on Quadrilateral Meshes





Yellow, Green and Red vertices are with topological valence 3, 5 and 6 respectively.

Singularities on Quadrilateral Meshes



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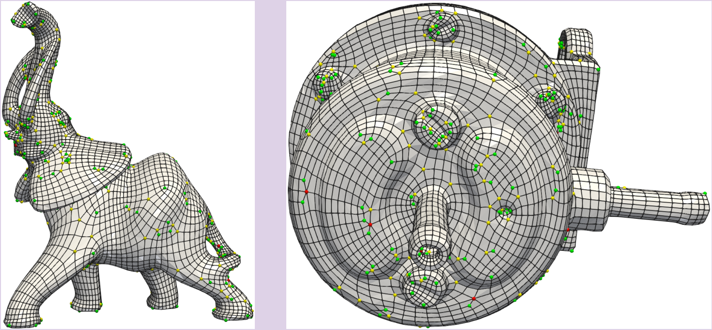
Singularities on Quadrilateral Meshes



Yellow, Green and Red vertices are with topological valence 3, 5 and 6 respectively.

Metric Holonomy Condition

Quad-Mesh Metric

Definition (Quad-Metric)

Given a quad-mesh $\mathscr{Q}$, each face is treated as the unit planar square, this will define a Riemannian metric, the so-called quad-mesh metric $\mathbf{g}_{\mathscr{Q}}$, which is a flat metric with cone singularities.

Theorem (Quad-Mesh Metric Conditions)

Given a quad-mesh $\mathscr{Q}$, the induced quad-mesh metric is $\mathbf{g}_{\mathscr{Q}}$, which satisfies the following four conditions:

1. *Gauss-Bonnet condition;*
2. *Holonomy condition;*
3. *Boundary Alignment condition;*
4. *Finite geodesic trajectory condition.*

Gauss-Bonnet Condition

Definition (Curvature)

Given a quad-mesh $\mathscr{Q}$, for each vertex v_i, the curvature is defined as

$$K(v) = \begin{cases} \frac{\pi}{2}(4-k(v)) & v \notin \partial\mathscr{Q} \\ \frac{\pi}{2}(2-k(v)) & v \in \partial\mathscr{Q} \end{cases}$$

where $k(v)$ is the topological valence of v, i.e. the number of faces adjacent to v.

Theorem (Gauss-Bonnet)

Given a quad-mesh $\mathscr{Q}$, the induced metric is $\mathbf{g}_{\mathscr{Q}}$, the total curvature satisfies

$$\sum_{v_i \in \partial\mathscr{Q}} K(v_i) + \sum_{v_i \notin \partial\mathscr{Q}} K(v_i) = 2\pi\chi(\mathscr{Q}).$$

Namely

Holonomy Condition

Definition (Holonomy)

Given a quad-mesh $\mathcal{Q}$, the induced flat metric is $\mathbf{g}_{\mathcal{Q}}$, the set of singular vertices is $S_{\mathcal{Q}}$. Suppose $\gamma : [0,1] \to \mathcal{Q} \setminus S_{\mathcal{Q}}$ is a closed loop not through singularities, choose a tangent vector $\mathbf{v}(0) \in T_{\gamma(0)}\mathcal{Q}$, parallel transport $\mathbf{v}(0)$ along $\gamma(t)$ to obtain $\mathbf{v}(t)$. The rotation angle from $\mathbf{v}(0)$ to $\mathbf{v}(1)$ in $T_{\gamma(0)}\mathcal{Q}$ is the holonomy of γ, denoted as $\rho(\gamma)$.

Because $\mathbf{g}_{\mathcal{Q}}$ is flat on $\mathcal{Q} \setminus S_{\mathcal{Q}}$, if γ_1 is homotopic to γ_2, then $\rho(\gamma_1) = \rho(\gamma_2)$. Therefore, holonomy is a homomorphism from the fundamental group to $\mathbb{S}^1$,

$$\lambda : \pi_1(\mathcal{Q} \setminus S_{\mathcal{Q}}) \to \mathbb{S}^1.$$

Face Loop

Definition (face path)

A sequence of faces, $\{f_0, f_1, \cdots, f_n\}$, such that f_i and f_{i+1} share an edge. If f_0 equals to f_n, then the face path is called a face loop.

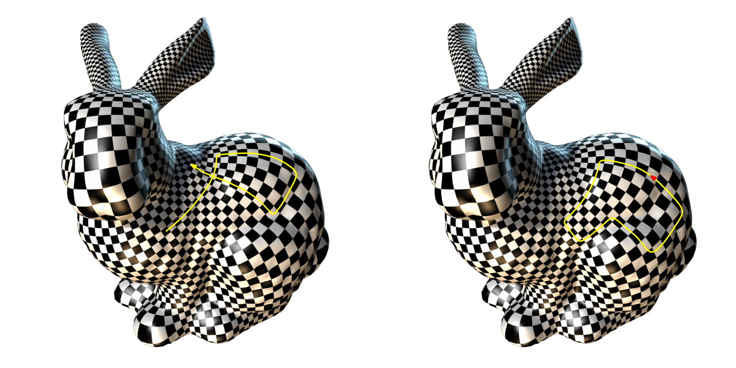


Figure: A face path and a face loop.

Fundamental Group

Definition (Fundamental Group)

Given a quad-mesh $\mathcal{Q}$ with singularities $S_{\mathcal{Q}}$, fix a base face σ_0, the homotopy classes of face loops through σ_0 form the fundamental group, denoted as $\pi_1(\mathcal{Q} - S_{\mathcal{Q}}, \sigma_0)$.

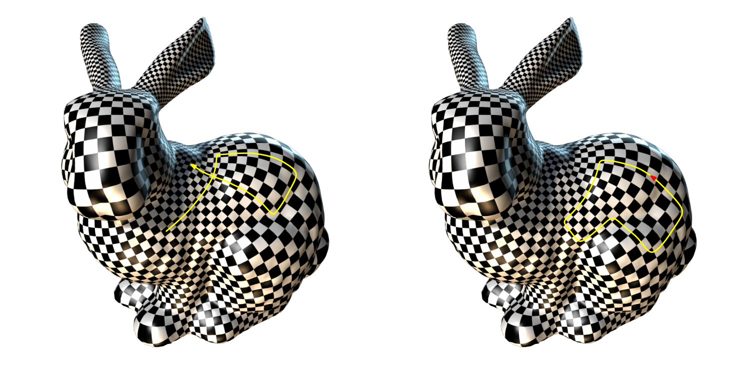
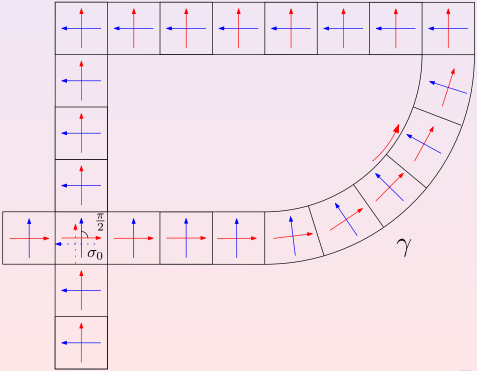


Figure: A face path and a face loop.

Holonomy

Definition (Holonomy of a loop)

Given a face loop γ through σ_0, fix a frame on σ_0, parallel transport the frame along γ. When we return to σ_0, the frame is rotated by an angle $k\pi/2$, which is called the holonomy of γ, and denoted as $\langle\gamma\rangle$.



Holonomy Condition

Theorem (Holonomy)

Given a quad-mesh $\mathscr{Q}$ with induced metric $\mathbf{g}_{\mathscr{Q}}$, the holonomy homomorphism is

$$\lambda : \pi_1(\mathscr{Q} \setminus S_{\mathscr{Q}}) \to \mathbb{S}^1,$$

then the holonomy group is a subgroup of rotation group

$$\lambda(\pi_1(\mathscr{Q} \setminus S_{\mathscr{Q}})) \subset \mathscr{R} = \{e^{i\frac{k\pi}{2}}, k = 0, 1, 2, 3\}.$$

Boundary Alignment Condition

Given a flat cone metric with satisfying the holonomy condition, one can define a global cross field by parallel transportation, which gives the stream lines.

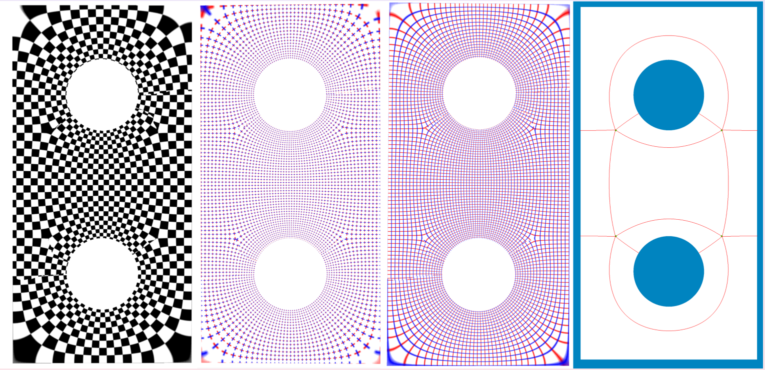


Figure: Quad-mesh with 4 saddle points.

Definition (Boundary Alignment Condition)

Given a quad-mesh $\mathcal{Q}$, with induced metric $\mathbf{g}_{\mathcal{Q}}$, one can define a global cross field by parallel transportation, which is aligned with the boundaries.

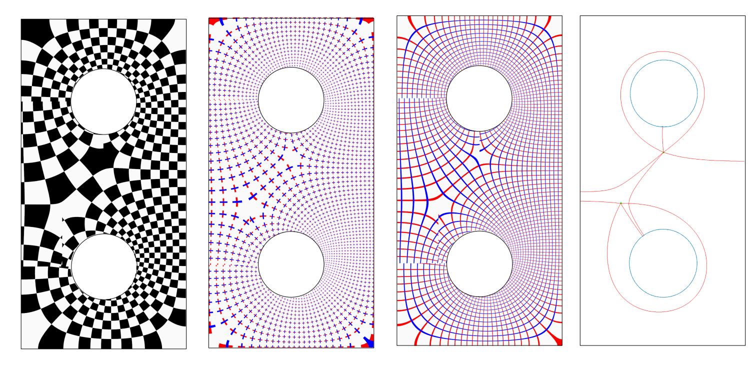


Figure: Cross field is mis-aligned with the inner boundaries.

Finite Geodesic Trajectory Condition

Definition (Finite Geodesic Trajectory Condition)

The stream lines parallel to the cross field are finite geodesic loops. This is the finite geodesic lamination condition.

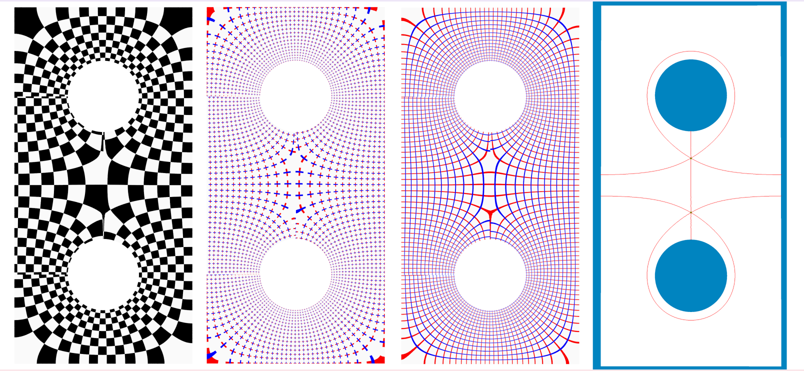


Figure: Finite geodesic trajectory condition.

Riemann Surface Theory

Riemann Surface

Definition (Riemann Surface)

Suppose M is a topological surface, with a complex atlas $\{(U_\alpha, \varphi_\alpha)\}$, where $\varphi_\alpha : U_\alpha \to \mathbb{C}$, such that all coordinate transition functions

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$$

are biholomorphic, then M is called a Riemann surface.

Definition (Meromorphic Function)

Suppose $f : M \to \mathbb{C} \cup \{\infty\}$ is a complex function defined on the Riemann surface M. If for each point $p \in M$, there is a neighborhood $U(p)$ of p with local parameter $z(p) = 0$, f has Laurent expansion

$$f(z) = \sum_{i=k}^{\infty} a_i z^i,$$

then f is called a meromorphic function.

Meromorphic Differential

Definition (Meromorphic Differential)

Given a Riemann surface $(M, \{z_\alpha\})$, ω is a meromorphic differential of order n, if it has local representation,

$$\omega = f_\alpha(z_\alpha)(dz_\alpha)^n,$$

where $f_\alpha(z_\alpha)$ is a meromorphic function, n is an integer; if $f_\alpha(z_\alpha)$ is a holomorphic function, then ω is called a holomorphic differential of order n.

Zeros and Poles

Definition (Zeros and Poles)

Suppose $f : M \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function. For each point p, there is a neighborhood $U(p)$ of p with local parameter $z(p) = 0$, f has Laurent expansion

$$f(z) = \sum_{i=k}^{\infty} a_i z^i,$$

if $k > 0$, then p is a zero with order k; if $k = 0$, then p is a regular point; if $k < 0$, then p is a pole with order k. The assignment of p with respect to f is denoted as $v_p(f) = k$.

The zeros and poles of a meromorphic differential are defined in the similar way.

Divisor

Definition (Divisor)

The Abelian group freely generated by points on a Riemann surface is called the divisor group, every element is called a divisor, which has the form $D = \sum_p n_p p$. The degree of a divisor is defined as $deg(D) = \sum_p n_p$. Suppose $D_1 = \sum_p n_p p$, $D_2 = \sum_p m_p p$, then $D_1 \pm D_2 = \sum_p (n_p \pm m_p) p$; $D_1 \leq D_2$ if and only if for all p, $n_p \leq m_p$.

Definition (Meromorphic Function Divisor)

Given a meromorphic funciton f defined on a Riemann surface S, its divisor is defined as $(f) = \sum_p v_p(f) p$, where $v_p(f)$ is the assignment of p with respect to f.

The divisor of a meromorphic function is called a principle divisor. The divisor of a meromorphic differential is defined in the similar way.

Principle Divisor

Theorem

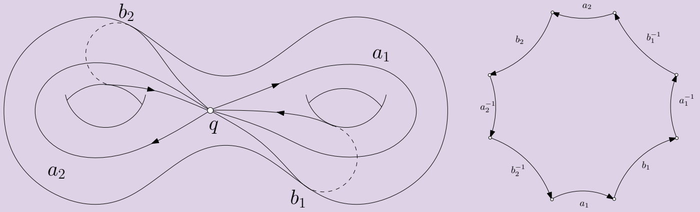
Suppose M is a compact Riemann surface with genus g, f is a meromorphic function, then

$$\deg((f)) = 0,$$

ω is a meromorphic differential, then

$$\deg((\omega)) = 2g - 2.$$

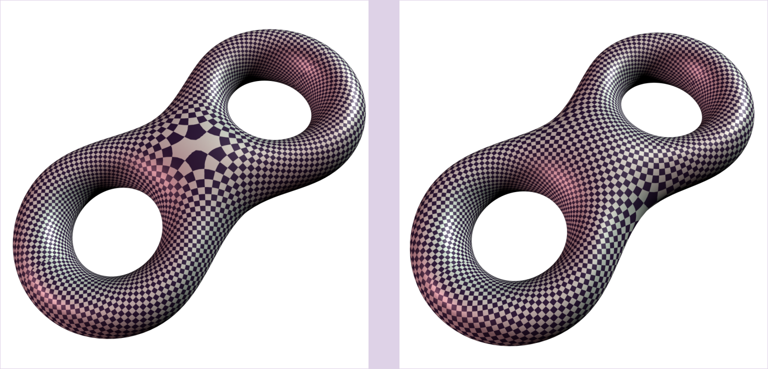
Canonical Fundamental Group Generators



Algebraic intersection numbers satisfy the conditions:

$$a_i \cdot b_j = \delta_{ij}, a_i \cdot a_j = 0, b_i \cdot b_j = 0.$$

Holomorphic Differential Group Basis



The holomorphic one-form basis $\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$ satisfy the dual condition

$$\int_{a_j} \varphi_i = \delta_{ij}.$$

Period Matrix

Definition (Period Matrix)

Suppose M is a compact Riemann surface of genus g, with canonical fundamental group basis

$$\{a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g\}$$

and holomorphic one form basis

$$\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$$

The period matrix is defined as $[A, B]$

$$A = \left(\int_{a_j} \varphi_i\right), B = \left(\int_{b_j} \varphi_i\right).$$

Matrix B is symmetric, $Img(B)$ is positive definite.

Jacobi Variety

Definition (Jacobi Variety)

Suppose the period matrix

$$A = (A_1, A_2, \cdots, A_g), \quad B = (B_1, B_2, \cdots, B_g),$$

the lattice Γ is

$$\Gamma = \left\{ \sum_{i=1}^{g} \alpha_i A_i + \sum_{j=1}^{g} \beta_j B_j \right\},$$

the Jacobi variety of M is defined as

$$J(M) = \mathbb{C}^g / \Gamma.$$

Jacobi Map

Definition (Jacobi Map)

Given a compact Riemann surface M, choose a set of canonical fundamental group generators $\{a_1,\cdots,a_g,b_1,\cdots,b_g\}$, and obtain a fundamental domain Ω,

$$\partial\Omega = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

choose a base point p_0, the Jacobi map $\mu : M \to J(M)$ is defined as follows: for any point $p \in M$, choose a path γ from p_0 to p inside Ω,

$$\mu(p) = \left(\int_\gamma \varphi_1, \int_\gamma \varphi_2, \cdots, \int_\gamma \varphi_g\right)^T.$$

Theorem (Abel)

Suppose M is a compact Riemann surface with genus g, D is a divisor, $\deg(D) = 0$. D is principle if and only if

$$\mu(D) = 0 \quad \text{in } J(M).$$

Quad-Mesh Abel condition

Divisor

Definition (Divisor)

The Abelian group freely generated by points on a Riemann surface is called the divisor group, every element is called a divisor, which has the form $D = \sum_p n_p p$. The degree of a divisor is defined as $deg(D) = \sum_p n_p$. Suppose $D_1 = \sum_p n_p p$, $D_2 = \sum_p m_p p$, then $D_1 \pm D_2 = \sum_p (n_p \pm m_p) p$; $D_1 \leq D_2$ if and only if for all p, $n_p \leq m_p$.

Definition (Quad-Mesh Divisor)

Suppose Q is a closed quadrilateral mesh, then Q induces a divisor

$$D_Q = \sum_{v_i \in Q} (k(v_i) - 4) v_i,$$

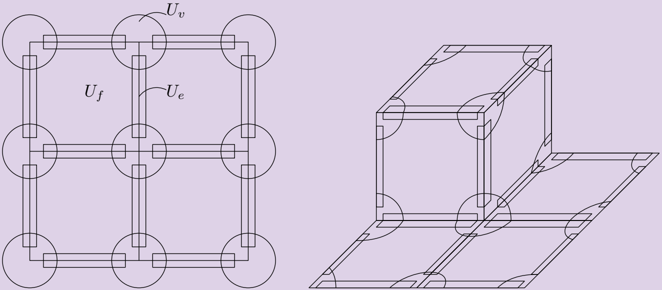
where v_i is a vertex with valence $k(v_i)$.

Quad-Mesh vs. Riemann Surface

Theorem (conformal structure induced by quad-mesh)

Suppose Q is a closed quadrilateral mesh, then Q induces a conformal structure and can be treated as a Riemann surface M_Q.

Proof.



(a) conformal atlas (b) singularities

Quad-Mesh Meromorphic Differential

Theorem (Quad-Mesh Meromorphic Differential)

Suppose Q is a closed quadrilateral mesh, then Q induces meromorphic quartic differential.

Proof.

On each face f, define dz_f, $\omega_Q = (dz_f)^4$; vertex face transition

$$z_v^{\frac{k}{4}} = e^{i\frac{n\pi}{2}} z_f + \frac{1}{2}(\pm 1 \pm i) \tag{1}$$

where k is the vertex valence, therefore

$$\left(\frac{k}{4}\right)^4 z_v^{k-4}(dz_v)^4 = (dz_f)^4 = \omega_Q. \tag{2}$$

$\square$

Quad-Mesh Abel Condition

Theorem (Gu-Lei, Quad-Mesh Abel Condition)

Suppose Q is a closed quadrilateral mesh, then for any holomorphic differential φ

$$\mu(D_Q - 4(\varphi)) = 0 \quad \textit{in } J(M_Q). \tag{3}$$

Inverse Proposition

Inversely, given a divisor D satisfies the Abel condition Eqn. 3, then there is a meromorphic quartic differential ω, such that $(\omega) = D$.

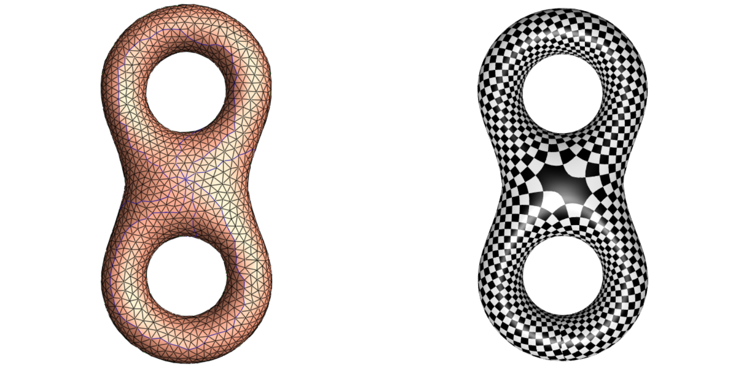
Holomorphic 1-form

Theoretic Foundation

Holomorphic 1-forms are computed based on Hodge theory, so that each cohomological class has a unique harmonic form.

Holomorphic One-form

Algorithm Input and Output



(a). input (b). output

Holomorphic One-form

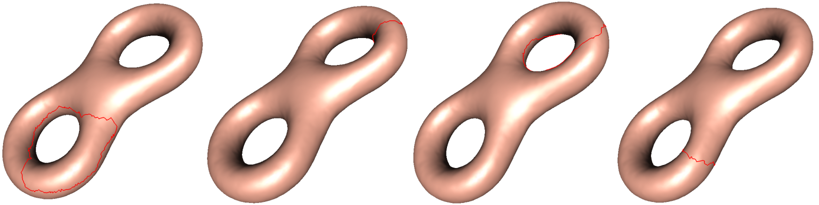
Computational Algorithm

Input: A compact, closed triangular polyhedron mesh M;
Output:A set of basis of the holomorphic one-form group, $\{\varphi_1, \varphi_2, \cdots, \varphi_{2g}\}$

1. Compute a basis of homology group $H_1(M, \mathbb{Z})$, $\{\gamma_1, \gamma_2, \dots, \gamma_{2g}\}$;
2. Compute a dual basis of cohomology group $H^1(M, \mathbb{R})$, $\{\tau_1, \tau_2, \cdots, \tau_{2g}\}$;
3. Diffuse the cohomology group basis to harmonic 1-forms, $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$;
4. Compute the conjugate harmonic 1-form ${}^*\omega_i$ for each ω_i, using Hodge star operator;
5. Construct a basis of holomorphic 1-form,

$$\left\{\omega_1 + \sqrt{-1}^*\omega_1, \omega_2 + \sqrt{-1}^*\omega_2, \cdots, \omega_{2g} + \sqrt{-1}^*\omega_{2g}\right\}$$

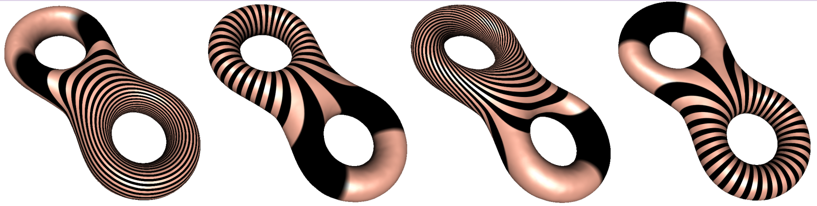
Homology Group Basis



Compute a CW-cell decomposition, compute the generators of the 1-skeleton.

Holomorphic One-form

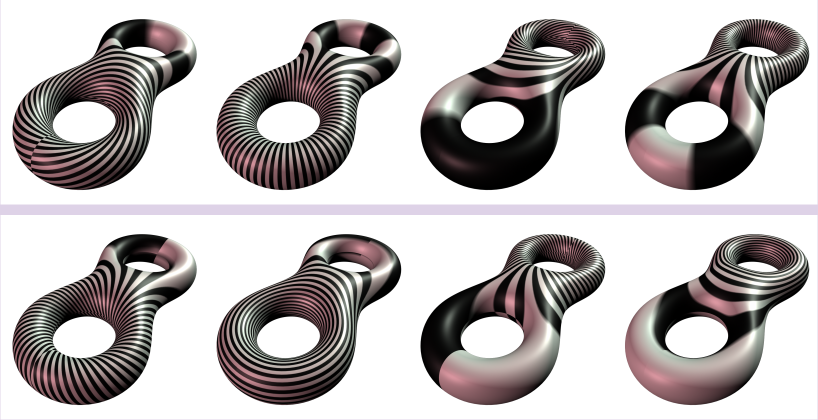
Harmonic 1-form group basis



Cut the surface along the homology group base curves, set the dual cohomology group basis, solve Poisson equations to obtain harmonic 1-form group basis.

Holomorphic One-form

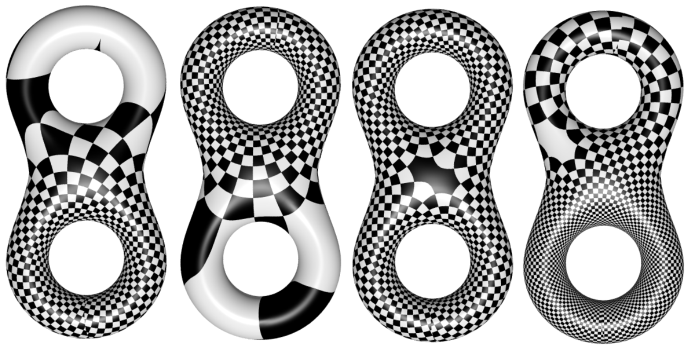
Conjugate Harmonic One-form



Compute the dual vector field for each harmonic 1-form, rotate the vector field about the normal for $\pi/2$ angle.

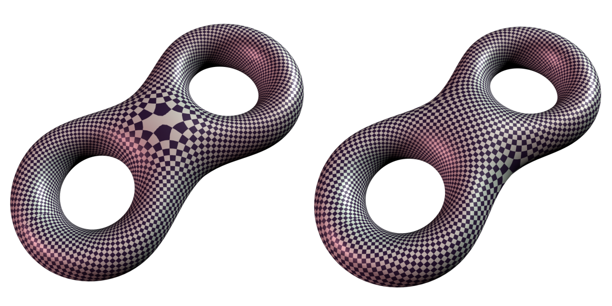
Holomorphic One-form

Holomorphic 1-form group basis



Holomorphic One-form

holomorphic 1-form construction by linear combination



We can compute the period matrix, and the Abel-Jacobi map

$$A = \left(\int_{a_j} \varphi_i\right), B = \left(\int_{b_j} \varphi_i\right), \quad \mu(p) = \left(\int_\gamma \varphi_1, \int_\gamma \varphi_2, \cdots, \int_\gamma \varphi_g\right)^T.$$

Holomorphic Quadratic Differential

Holomorphic One-form

Computational Strategy

- Prof. S. Gortler and D. THurston, “Discrete Quadratic Differentials”$≫
- Holomorphic quadratic differentials are computed based on R. Scheon’s graph-valued harmonic map, so that the Hopf differential of the harmonic map gives the holonomorphic quadratic differential (CMAME 2017).

Figure: A finite measured foliation on a genus two surface.

Foliation

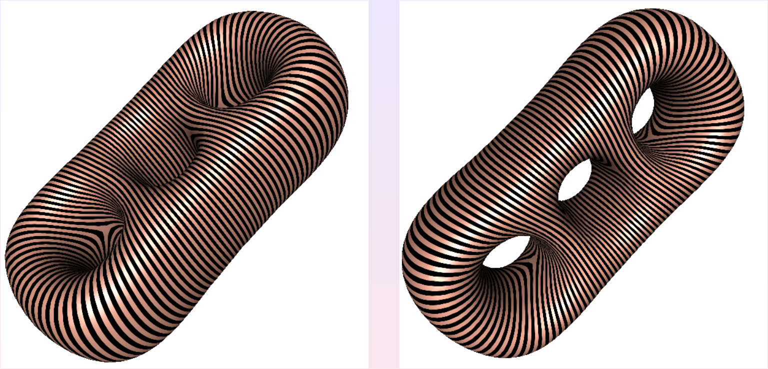


Figure: A finite measured foliation on a genus three surface.

Measured Foliation

Definition (Measured Foliation)

Let S be a compact Riemann surface of genus $g > 1$. A C^k measured foliation on S with singularities $z_1, \dots, z_l$ of order $k_1, \dots, k_l$ respectively is given by an open covering $\{U_i\}$ of $S - \{z_1, \dots, z_l\}$ and open sets $V_1, \dots, V_l$ around $z_1, \dots, z_l$ respectively along with C^k real valued functions v_i defined on U_i s.t.

1. $|dv_i| = |dv_j|$ on $U_i \cap U_j$
2. $|dv_i| = |\mathrm{Im}(z - z_j)^{k_j/2} dz|$ on $U_i \cap V_j$.

The kernels $\ker dv_i$ define a C^{k-1} line field on S which integrates to give a foliation $\mathscr{F}$ on $S - \{z_1, \dots, z_l\}$, with $k_j + 2$ pronged singularity at z_j. Moreover, given an arc $\gamma \subset S$, we have a well-defined measure $\mu(\gamma)$ given by $\mu(\gamma) = |\int_\gamma dv|$, where $|dv|$ is defined by $|dv|_{U_i} = |dv_i|$.

Measured Foliation

Figure: Finite measured foliations on a genus three surface.

Measured Foliation

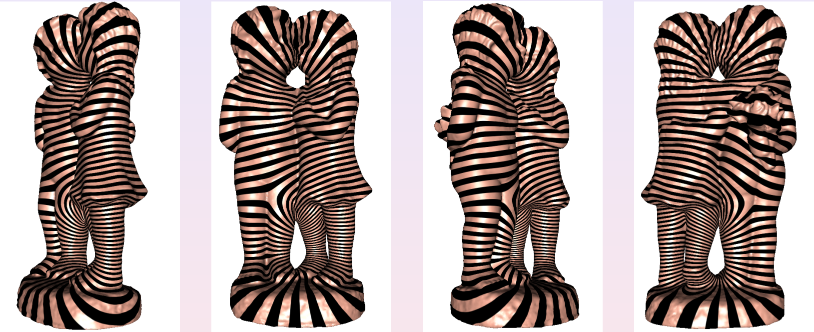


Figure: Holomorphic quadratic differentials on a genus three surface.

Whitehead Move

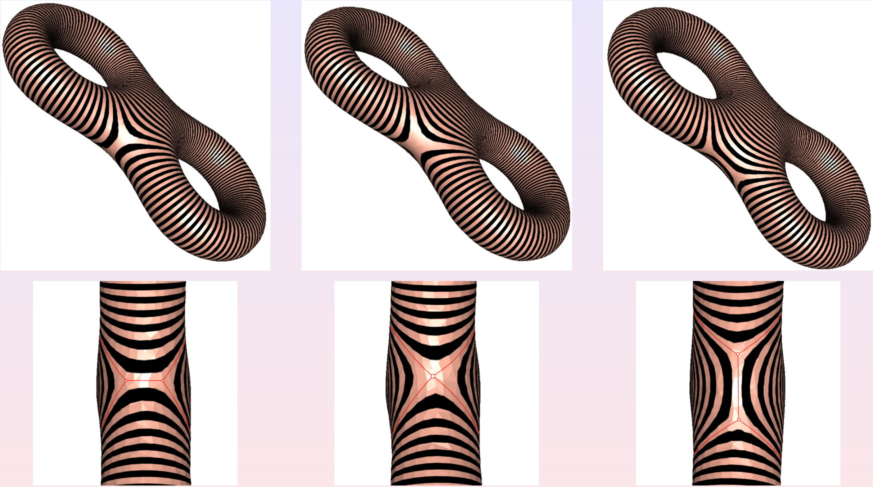


Figure: Equivalent measured foliations and Whitehead moves.

Equivalence

Two measured foliations $(\mathscr{F}, \mu)$ and $(\mathscr{G}, \nu)$ are said to be equivalent if after some Whitehead moves on $\mathscr{F}$ and $\mathscr{G}$, there is a self-homeomorphism of S isomorphic to the identity, which takes $\mathscr{F}$ to $\mathscr{G}$, and μ to ν.

Holomorphic Quadratic Differentials

Definition (Holomorphic Quadratic Differentials)

Suppose S is a Riemann surface. Let Φ be a complex differential form, such that on each local chart with the local complex parameter $\{z_\alpha\}$,

$$\Phi = \varphi_\alpha(z_\alpha)dz_\alpha^2,$$

where $\varphi_\alpha(z_\alpha)$ is a holomorphic function.

- A holomorphic quadratic differentials on a genus zero closed surface must be 0.
- The linear space of all holomorphic quadratic differentials is 1 complex dimensional, where the genus $g = 1$.
- The linear space of all holomorphic quadratic differentials is $3g-3$ complex dimensional, where the genus $g > 1$.

Zeros

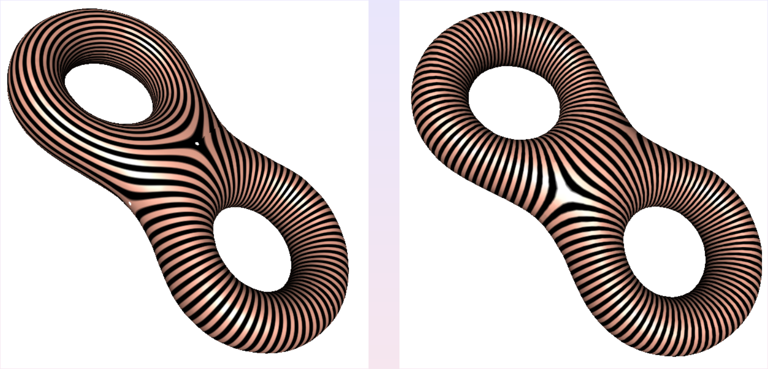


Figure: Holomorphic quadratic forms on the genus two surface.

Definition (Zeros)

A point $z_i \in S$ is called a *zero* of Φ, if $\varphi(z_i)$ vanishes. A holomorphic quadratic differential has $4g-4$ zeros.

Natural Coordinates

Definition (Natural Coordinates)

For any point away from zero, we can define a local coordinates

$$\zeta(p) := \int^{p} \sqrt{\varphi(z)}dz. \tag{4}$$

which is the so-called *natural coordinates* induced by Φ.

The curves with constant real natural coordinates are called the *vertical trajectories*, with constant imaginary natural coordinates *horizontal trajectories*. The trajectories through the zeros are called the *critical trajectories*.

Natural Coordinates

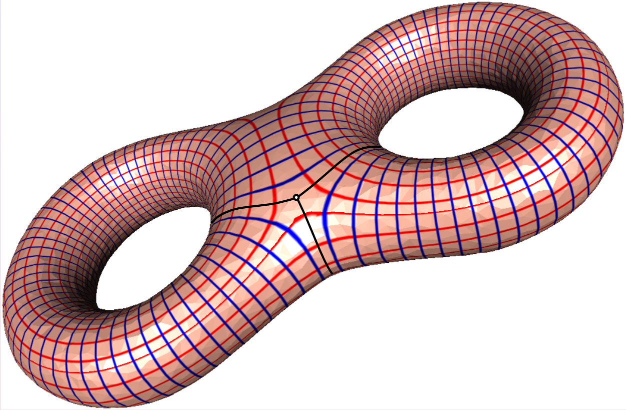
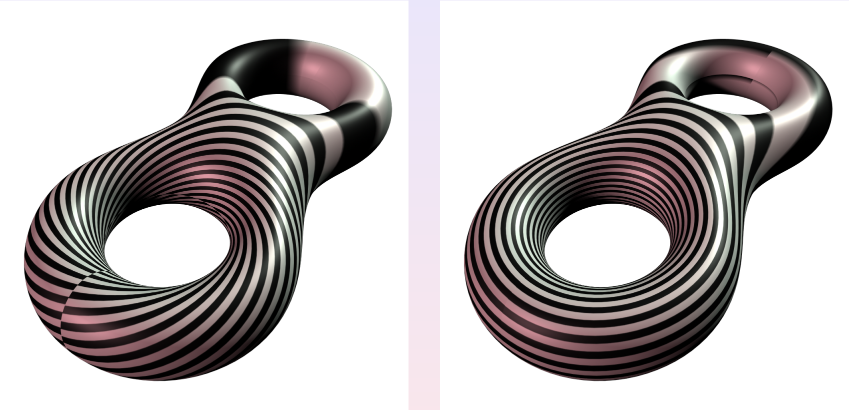


Figure: Trajectories of a holomorphic quadratic differential, blue - horizontal, red - vertical, black -critical trajectory.

Strebel Differential



(a) non-Strebel (b) Strebel

Figure: A non-Strebel (a) and a Strebel differential (b).

Strebel Differential

Definition (Strebel)

Given a holomorphic quadratic differential Φ on a Riemann surface S, if all of its horizontal trajectories are finite, then Φ is called a Strebel differential.

A holomorphic quadratic differential Φ is Strebel, if and only if its critical horizontal trajectories form a finite graph. The horizontal trajectories of a holomorphic differential may be infinite spirals as in the left frame, or finite loops as in the right frame.

From Differential to Foliation

Given a holomorhic quadratic differential Φ on a Riemman surface S, it defines a measured foliation in the following way: Φ induces the natural coordinates ζ, the local measured foliations are given by

$$(\{\text{Im}\zeta = \text{const}\}, |d\text{Im}\zeta|), \tag{5}$$

then piece together to form a measured foliation known as the *horizontal measured foliation* of Φ. Similarly, the vertical measured foliation of Φ is given by

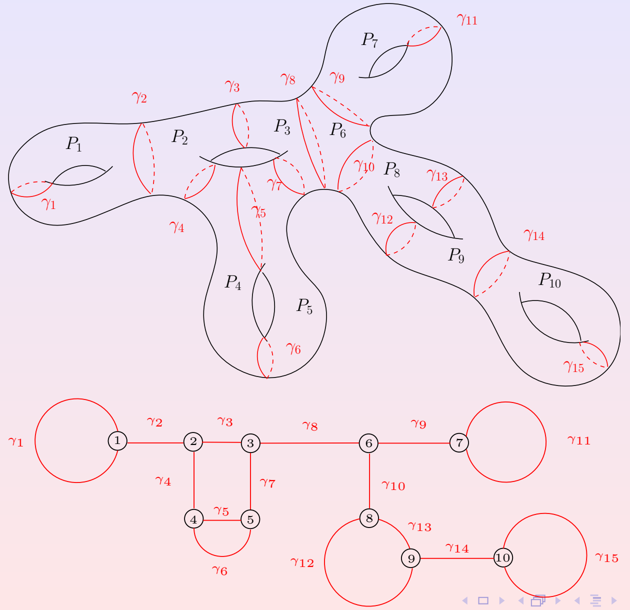
$$(\{\text{Re}\zeta = \text{const}\}, |d\text{Re}\zeta|). \tag{6}$$

Hubbard and Masure proved the following fundamental theorem connecting measured foliation and holomorphic quadratic differentials.

Theorem (Hubbard-Masur)

If $(\mathscr{F}, \mu)$ is a measured foliation on a compact Riemann surface S, then there is a unique holomorphic quadratic differential Φ on S whose horizontal foliation is equivalent to $(\mathscr{F}, \mu)$.

Pants Decomposition Graph



Pants Decomposition

Definition (Pants Decomposition)

Given a genus $g > 1$ closed surface S, a set of $3g-3$ disjoint simple loops, $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3}\}$ is called an *admissible curve system*. Γ segments S into $2g-2$ pairs of pants, $\{P_1, P_2, \cdots, P_{2g-2}\}$, this forms a *pants decomposition* of the surface.

Definition (Pants Decomposition Graph)

- Each pair of pants is represented as a node.
- Each simple loop is denoted by an edge. Suppose the simple loop γ_i connecting two pairs of pants P_j, P_k, then the arc of γ_i connects nodes of P_j and P_k. G is called the *pants decomposition graph*.

Existence of Strebel Differential

Theorem (Jenkins-Strebel)

Given non-intersecting simple loops $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3}\}$, and positive numbers $\{h_1, h_2, \cdots, h_{3g-3}\}$, there exists a unique holomorphic quadratic differential Φ, unique up to scaling, satisfying the following :

1. *The critical graph of Φ partition the surface into $3g-3$ cylinders, $\{C_1, C_2, \cdots, C_{3g-3}\}$, such that γ_k is the generator of C_k,*
2. *The modulus each cylinder $(C_k, |\Phi|)$ equals to λh_k, $k = 1, 2, \cdots, 3g-3$, for some positive constant λ.*

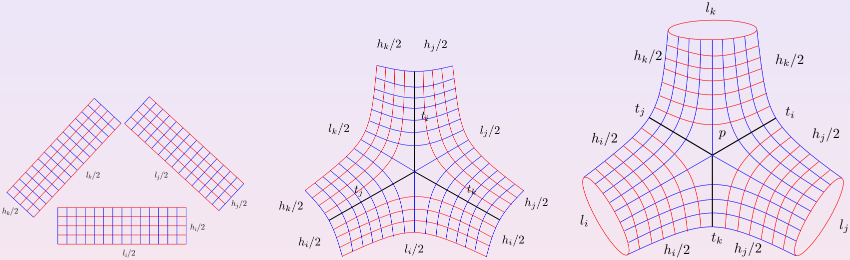
Poly-cylinder Surface

Given a Riemann surface S with genus $g > 1$, Φ is a Strebel differential, then the natural coordinates of Φ $\zeta : U \to \mathbb{C}$ induces a flat metric with cone singularities, denoted as $|\Phi|$,

1. The zeros of Φ become cone singularities, with cone angle 3π,
2. The critical graph of Φ partitions the surface into cylinders $\{C_1, C_2, \cdots, C_{3g-3}\}$, the generators of the cylinders are $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3}\}$,
3. The pants decomposition graph induced by Γ is denoted as G_Γ,
4. The heights of cylinder $(C_k, |\Phi|)$ is h_k,
5. The circumference of $(C_k, |\Phi|)$ is l_k,
6. The twisting angle of C_k is θ_k,

then $(S, |\Phi|)$ can be represented by $(G_\Gamma, \mathbf{h}, \mathbf{l}, \theta)$.

Poly-cylinder Surface

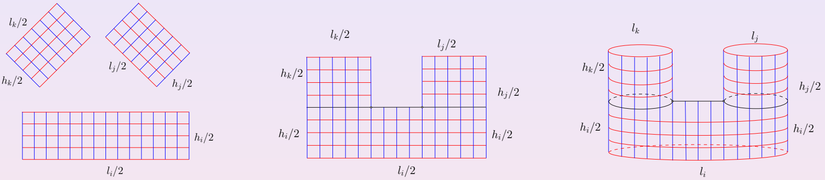


(a) 3 rectangles
$l_j + l_k > l_i$

(c) a hexagon

(e) a pair of pants (type I)

Figure: Flat cylindric surface model of $(S, |\Phi|)$.



(a) 3 rectangles $l_j + l_k < l_i$

(b) a hexagon

(f) a pair of pants (type II)

Figure: Flat cylindric surface model of $(S, |\Phi|)$.

Poly-cylinder Surface

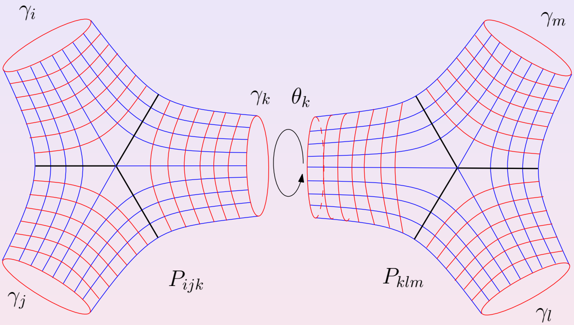
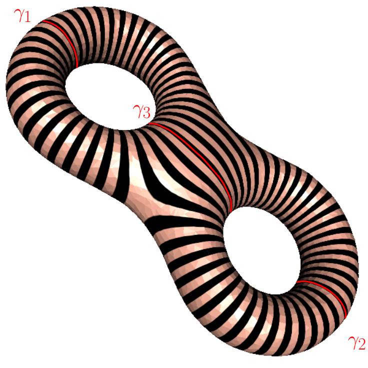
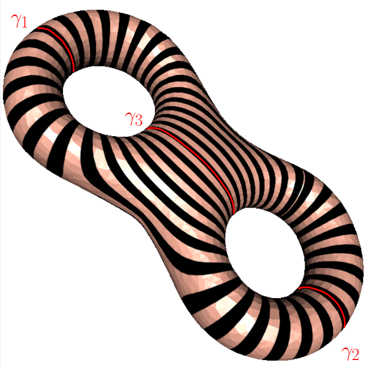


Figure: The twisting angle when gluing two pairs of pants.



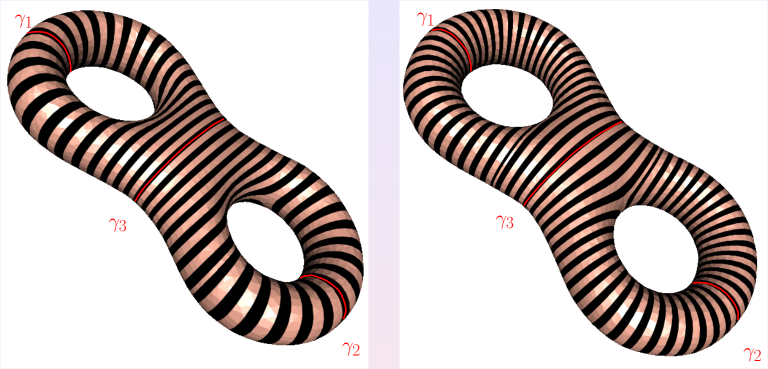
(a)



(b)

Figure: Strebel differentials on the genus two surface.

In the poly-cylinder surface model $(G_\Gamma, \mathbf{h}, \mathbf{l}, \theta)$, (l, θ) give a local coordinates of the Teichmüller space. The height function $\mathbf{h}$ changes.



(c) (d)

Figure: Strebel differentials on the genus two surface.

In the poly-cylinder surface model $(G_\Gamma, \mathbf{h}, \mathbf{l}, \theta)$, (l, θ) give a local coordinates of the Teichmüller space. The height function $\mathbf{h}$ changes.

Given a Strebel differential Φ, we obtain a poly-cylinder surface,

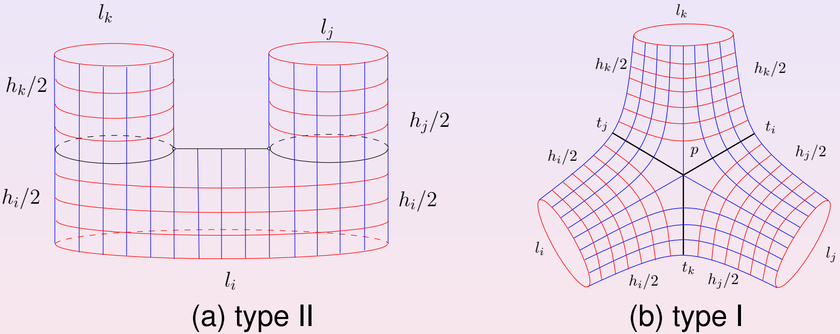


Figure: Change each pair of pants of type II to that of type I by a Whitehead move.

Differential to Quad-Mesh

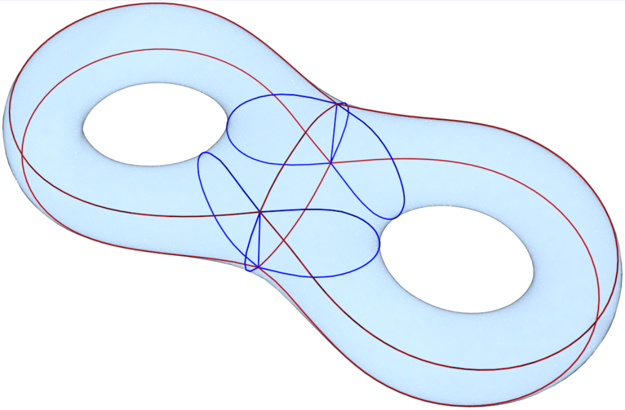


Figure: Divide each cylinder to two rectangles by connecting corresponding zeros on different boundary components; Construct an initial colorable quad-mesh;Subdivide.

Algorithmic Pipeline

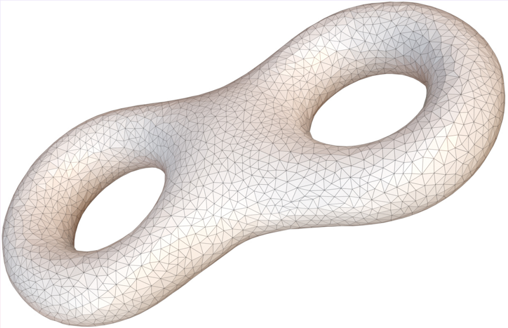
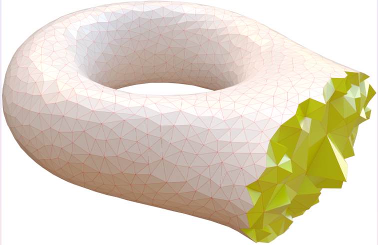


Figure: Input Surface.

Algorithmic Pipeline



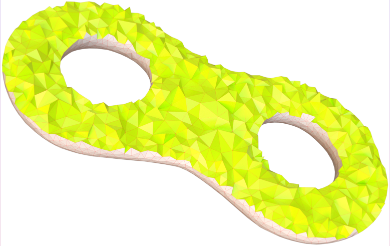


Figure: Tetrahedral meshing.

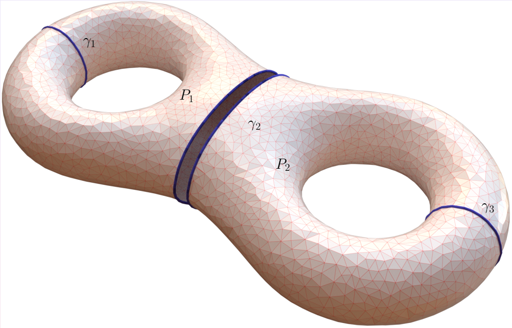


Figure: Admissible curve system, pants decomposition.

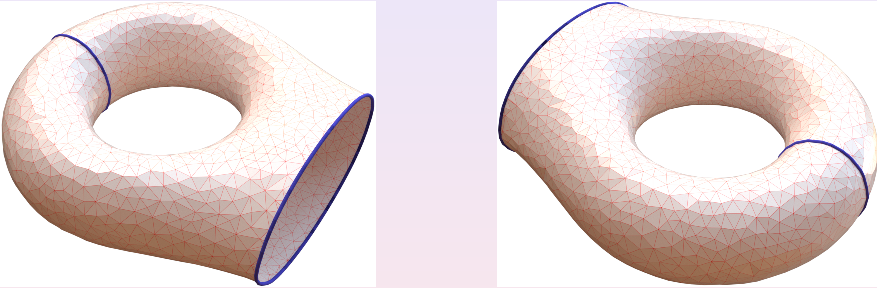


Figure: Two pairs of pants.

Algorithmic Pipeline

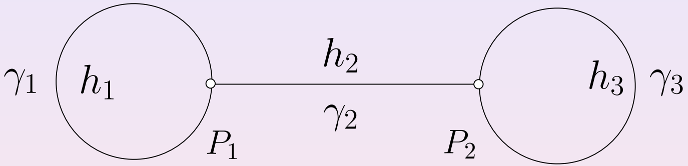


Figure: Pants decomposition graph.

Strebel Differentials

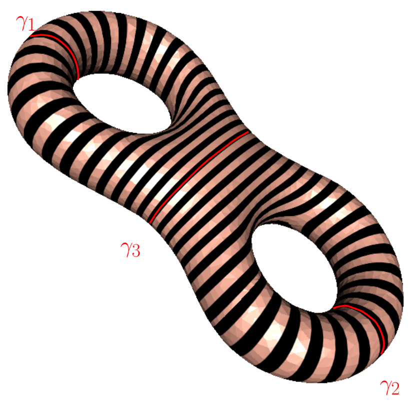


Figure: Holomorphic quadratic differential.

Algorithmic Pipeline

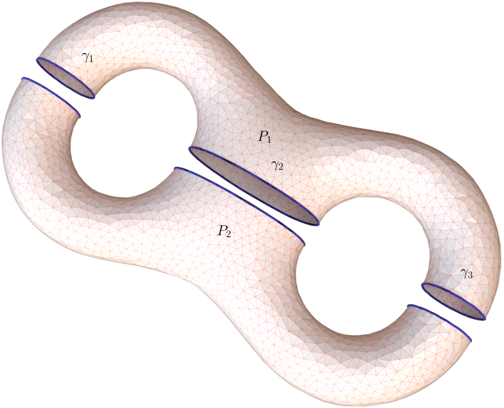


Figure: Admissible curve system, pants decomposition.

Algorithmic Pipeline

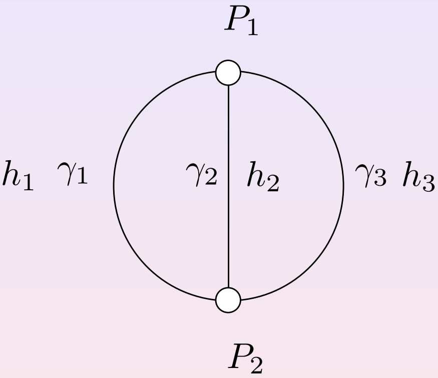


Figure: Pants decomposition graph.

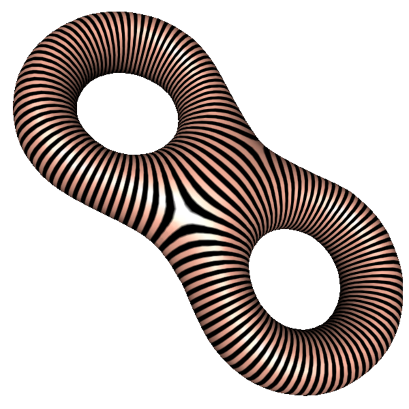


Figure: Holomorphic quadratic differential.

Algorithmic Pipeline

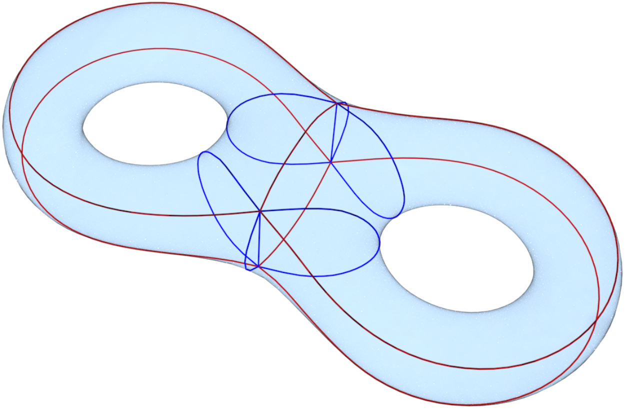


Figure: Critical horizontal trajectories and vertical trajectories.

Algorithmic Pipeline

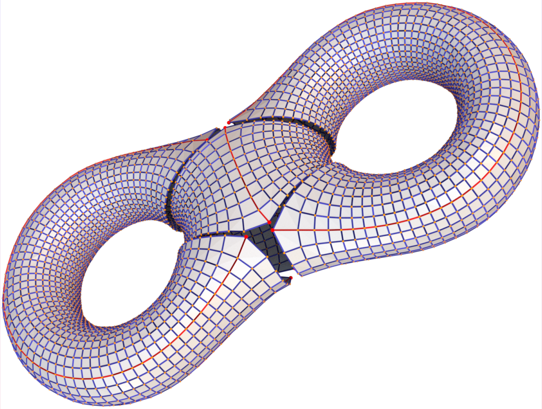


Figure: Cylindrical decomposition.

Algorithmic Pipeline

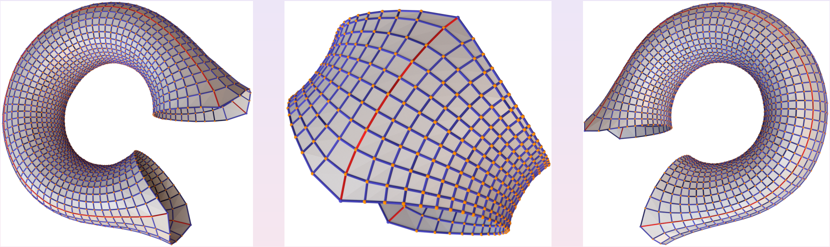


Figure: Cylindrical decomposition.

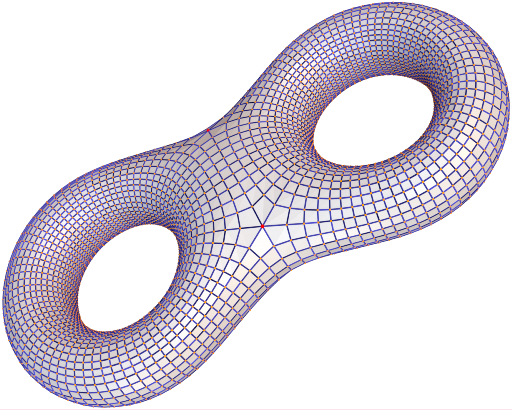
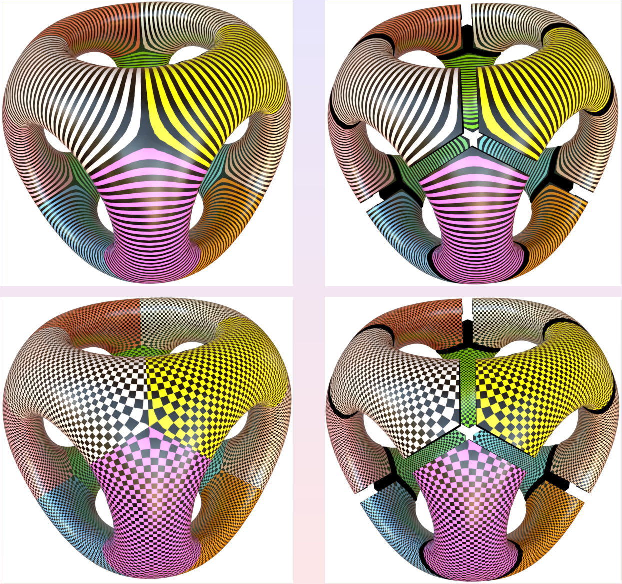


Figure: Colorable quadrilateral mesh.

Foliations



Foliations

Genus Three Model

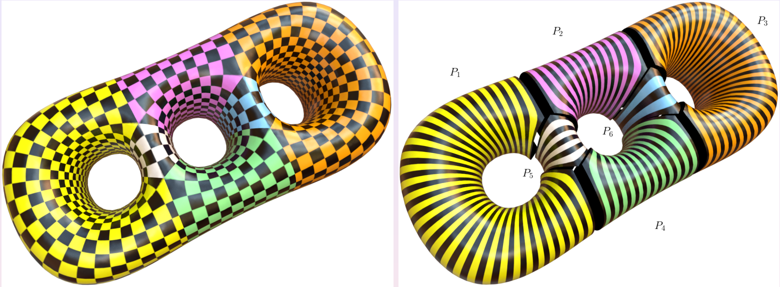


Figure: Holomorphic quadratic differential of a genus three surface.

Genus Three Model

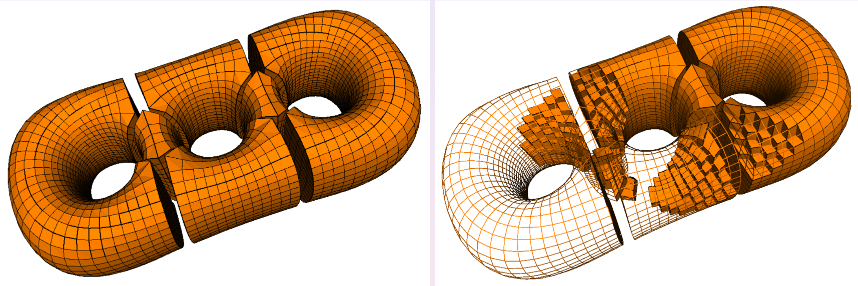


Figure: Hexahedral mesh of a genus three model.

Hex-Mesh

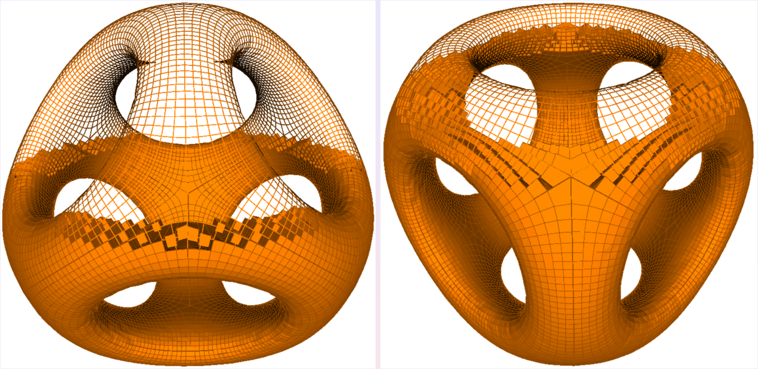


Figure: Hexahedral mesh of decocube model.

Meromorphic Quartic Differential

Meromorphic Quartic Differential

Theoretic Foundation

Meromorphic quartic differentials are computed based on Abel-Jacobi theorem and discrete surface Yamabe flow, such that the flat metric with the cone singularities at the divisor is the metric induced by the meromorphic differential.

Vertex Scaling

Definition (Vertex Scaling)

Two triangulated PL surface $(S, \mathcal{T}, d)$ and $(S, \mathcal{T}, d')$ are said to differ by a vertex scaling, if $\exists \lambda : V(\mathcal{T}) \to \mathbb{R}_{>0}$, such that $d' = \lambda_* d$ on $E(\mathcal{T}$, where

$$\lambda_* d(u,v) = \lambda(u)\lambda(v)d(u,v).$$

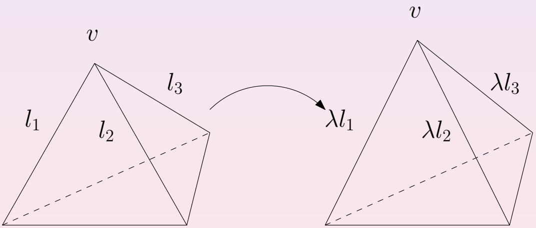


Figure: vertex scaling.

Discrete Conformal Equivalence

Definition (Gu-Luo-Sun-Wu)

Two PL metrics d, d' on a closed marked surface (S, V) are *discrete conformal*, if they are related by a sequence of two types of moves: vertex scaling and edge flip preserving Delaunay property.

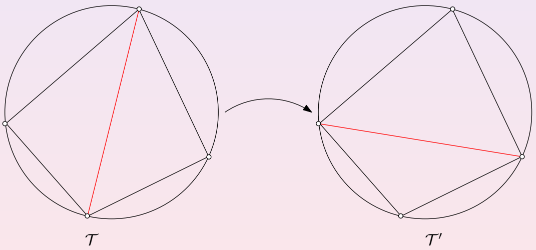


Figure: Edge flip, both triangulations are Delaunay.

Given a PL metric d on (S, V), produce a Delaunay triangulation $\mathscr{T}$ of (S, V),

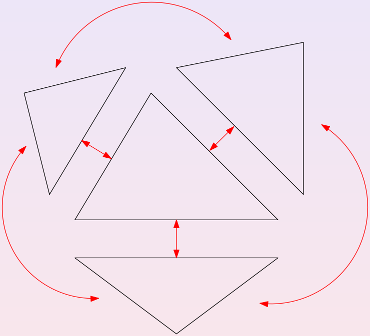
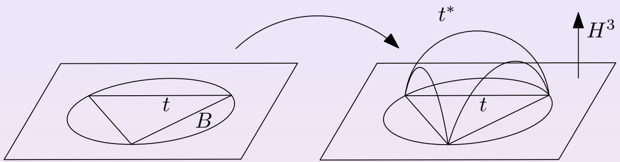
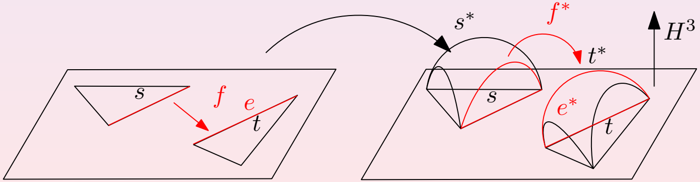


Figure: (S, V) with PL metric d, the triangulation is Delaunay.

Each face $t \in \mathscr{T}$ is associated an ideal hyperbolic triangle:

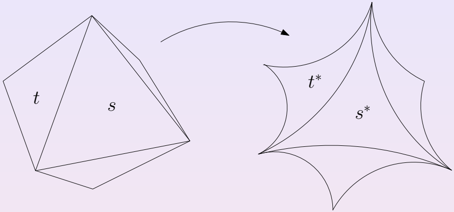


If $t, s \in \mathscr{T}$ glued by isometry f along e, then t^* and s^* are glued by the same f^* alonge e^*,



Discrete Conformal Equivalence

This induces a hyperbolic metric d^* on $S - V$.



Motivated by the important work of Bobenko-Pinkall-Springborn, equivalent to the previous defintion using vertex scaling and Delaunay condition.

Definition (Gu-Luo-Sun-Wu, JDG 2018)

Two PL metrics d_1 and d_2 on (S, V) are *discrete conformal* iff d_1^* and d_2^* are isometric by an isometry homotopic to identity on $S - V$.

Existence of the metric

Theorem (Gu-Luo-Sun-Wu)

Given a PL metric d on a closed marked surface (S,V), and curvature $K^*: V \to (-\infty, 2\pi)$, such that K satisfies the Gauss-Bonnet condition $\sum K(v) = 2\pi\chi(S)$, there there is a d^* discrete conformal to d, and d^* realizes the curvature K^*.

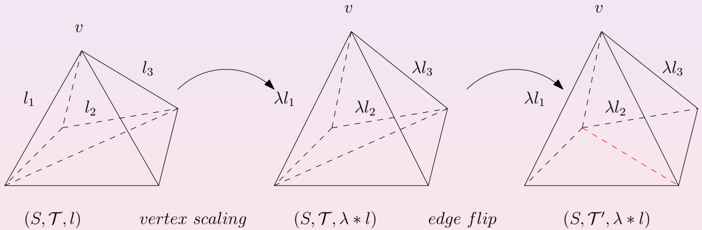


Figure: Discrete surface Yamabe flow.

Discrete Conformal Equivalence

Convex Optimization

Using Newton's method to minimize the following energy

$$\min_{\lambda} \int^{(\lambda_1,\lambda_2,\ldots,\lambda_n)} \sum_v (K^*(v) - K(v))\, d\log\lambda(v),$$

such that $\Pi_v \lambda(v) = 1$. During the optimization, keep the triangulation always to be Delaunay.

Algorithm for Meromorphic Quartic Differential

Input: A closed genus g triangle mesh M, and an initial divisor D;
Output: A meromorphic quartic differential ω, such that (ω) is close to D;

1. Compute 4 holomorphic 1-forms, $\varphi_1, \varphi_2, \cdots, \varphi_4$;
2. Check if the input divisor D satisfies the Gauss-Bonnet condition;
3. Optimize the positions of the zeros and poles, such that
$$\mu\left(D - \sum_{k=1}^{4}(\varphi_k)\right) \equiv 0 \mod \Gamma.$$
4. Set the target Gaussian curvatures at the zeros and poles in D, use discrete surface Yamabe flow to compute a flat metric;
5. Choose a horizontal direction, issue critical trajectories from the singularities to form a motograph;
6. The motorgraph divides the surface into topological rectangles, each of them is isometrically mapped to the plane using the flat metric.
7. The differentials $(dz)^4$ on the local chart are pulled back to obtain ω.

Bull Model

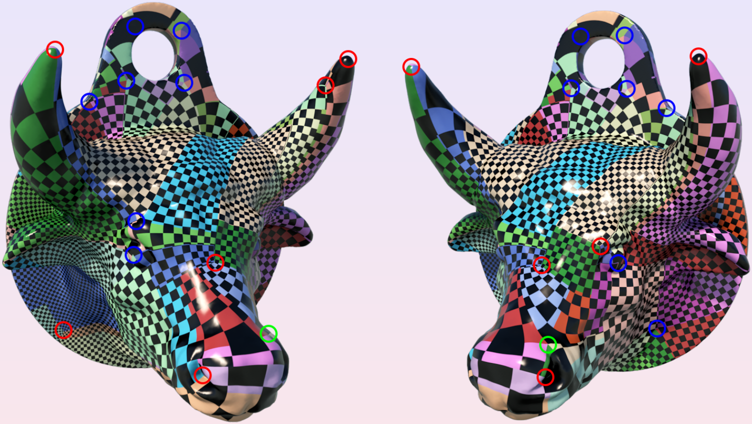


Figure: Meoromorphic quartic differential on a bull model.

Bull Model

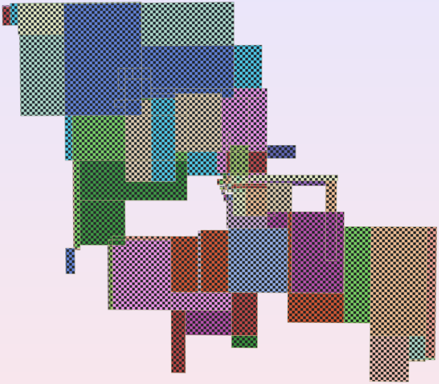


Figure: Meoromorphic quartic differential on a bull model.

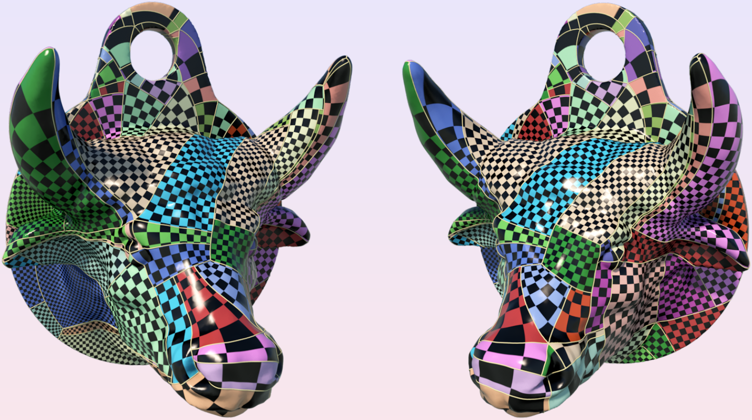


Figure: Meoromorphic quartic differential on a bull model.

Buddha Model

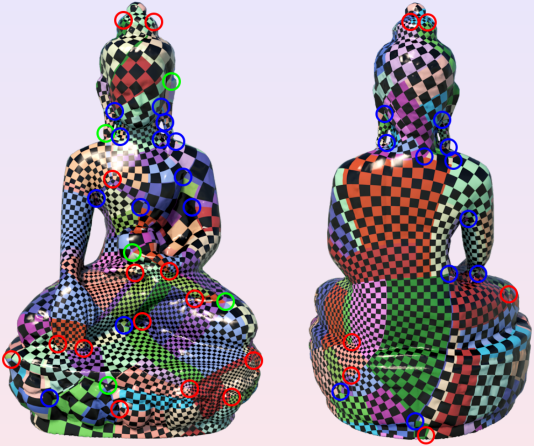


Figure: Meoromorphic quartic differential on a Buddha model.

Buddha Model

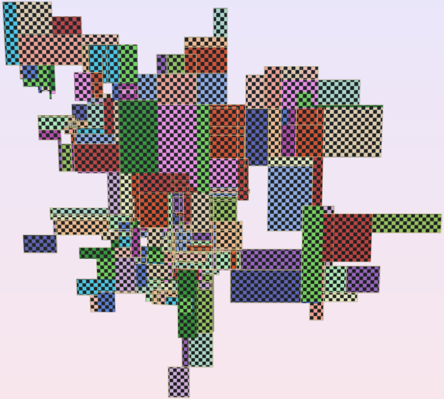


Figure: Meoromorphic quartic differential on a Buddha model.

Buddha Model

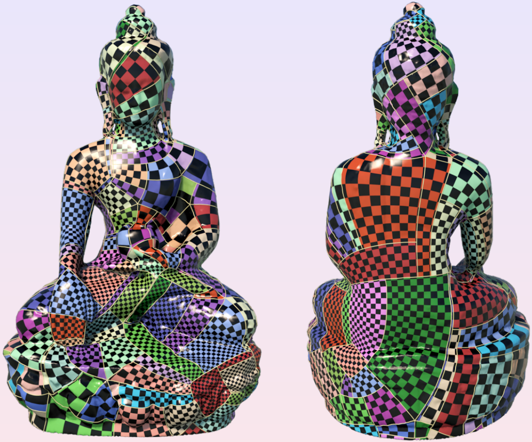


Figure: Meoromorphic quartic differential on a Buddha model.

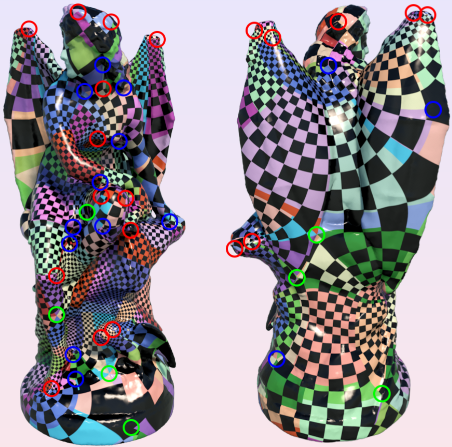


Figure: Meoromorphic quartic differential on a witch model.

Witch Model

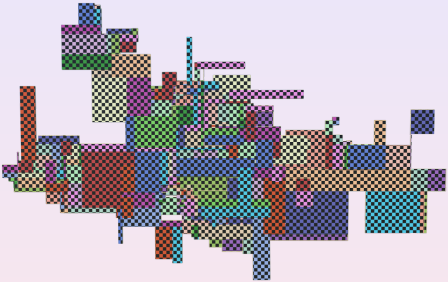


Figure: Meoromorphic quartic differential on a witch model.

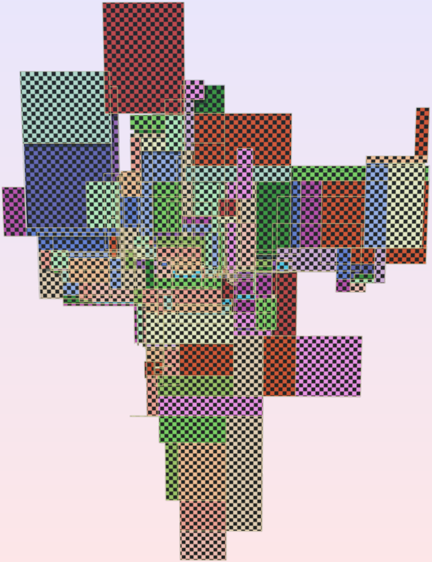
witch Model

Figure: Meoromorphic quartic differential on a witch model.

Dancer Model

Figure: Meoromorphic quartic differential on a dancer model.

Dancer Model



Dancer Model

Figure: Meoromorphic quartic differential on a dancer model.

Summary

- Build the connection between quad-meshes and meromorphic differentials.
- Holomorphic one-form based on Hodge theory;
- Holomorphic quadratic differential based on graph-valued harmonic maps;
- Meromorphic quartic differential based on Abel-Jacobi condition and discrete Yamabe flow.

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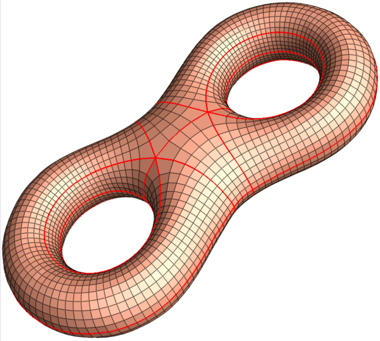
Future Direction

The Abel-Jacobi condition guarantees the holonomy condition. The finite trajectory condition needs to deform the conformal structure of the surface. Generally, meromorphic quartic differentials can produce T-Splines, which can satisfy the requirements in computational mechanics.

Thanks

For more information, please email to gu@cs.stonybrook.edu.



Thank you!